Year 12 proof

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Contents

1 The idea of proof

- A statement that has been proven is called a theorem. For example, Pythagoras' theorem.
- A statement that has yet to be proven is called a conjecture. However, there are cases that have gone against this like Fermat's last theorem which was famously proved by Andrew Wiles in 1995.
- A proof is a logical and structured argument to show that a mathematical statement is always true. It should be structured so that it follows a logical series of steps.
- It is standard to end a proof with what is referred to as a 'proof square' to say that you are finished proving your conjecture.

2 Proof by deduction

A method of proof that can be used to prove that a conjecture is true is deduction. This involves a logical argument as to why the conjecture must be true. This will often require you to use algebra.

2.1 Proofs with odd and even numbers

Example

Prove that the difference between the cube and square of an odd number is even.

Proof. First of all we need to think about what we need to do here and how the argument should be laid out. We need to prove that that the difference between the cube and square of ANY odd number is even. You cannot simply substitute a few odd numbers in and see that it works for those particular cases.

It was stated earlier that you often need to use algebra in proof by deduction. We can denote an odd number by $2n+1$ where n is an integer (a whole number). The expression $2n+1$ will always be an odd number no matter what integer n equals. For example, if $n = 4$, then the expression $2n+1$ would equal 9 which is odd.

The cube of $2n+1$ is $(2n+1)^3$ and the square of $2n+1$ is $(2n+1)^2$.

The difference between them is denoted by $(2n+1)^3 - (2n+1)^2$. Now we just need to simplify this expression by expanding brackets and cleaning it up. At the end we will have to make it obvious that the end result is always an even number.

$$\begin{aligned}
(2n+1)^3 - (2n+1)^2 &= (2n+1)(2n+1)(2n+1) - (2n+1(2n+1) \\
&= (4n^2+4n+1)(2n+1) - (4n^2+4n+1) \\
&= (8n^3+4n^2+8n^2+4n+2n+1) - (4n^2+4n+1) \\
&= (8n^3+12n^2+6n+1) - (4n^2+4n+1) \\
&= (8n^3+12n^2+6n+1) - 4n^2 - 4n - 1 \\
&= 8n^3+8n^2+2n \\
&= 2(4n^3+4n^2+n)
\end{aligned}$$

$2(4n^3+4n^2+n)$ is an even number as 2 multiplied by any whole number is even. So the difference between the cube and square of an odd number is even. □

Example

Given that n can be any integer such that $n > 1$, prove that $n^2 - n$ is never an odd number.

Proof. This is a slightly different proof to the last one. We are not told whether n is even or whether it is odd so we will need to consider two cases here. One where n is even and one where n is odd.

Case 1: n is even

If n is even then $n = 2m$ where m is a positive integer (remember that $n > 1$). Some students make a silly error and put $2n$. This makes no sense at all as n cannot equal $2n$ unless $n = 0$ but it does not here. You need to choose a different letter!

It then follows that $n^2 - n = (2m)^2 - 2m = 4m^2 - 2m = 2(2m^2 - m)$ which is even as 2 multiplied by any whole number is even. So $n^2 - n$ is not odd for when n is even.

Case 2: n is odd

If n is odd then $n = 2m + 1$ where m is a positive integer (remember that $n > 1$). Some students make a silly error and put $2n + 1$. This makes no sense at all as n cannot equal $2n + 1$ unless $n = -1$ but it does not here. You need to choose a different letter!

It then follows that

$$\begin{aligned} n^2 - n &= (2m+1)^2 - (2m+1) \\ &= (4m^2 + 4m + 1) - (2m + 1) \\ &= 4m^2 + 2m \\ &= 2(2m^2 + m) \end{aligned}$$

which is even as 2 multiplied by any whole number is even. So $n^2 - n$ is not odd for when n is odd.

Therefore, the expression $n^2 - n$ is never odd. □

2.2 Proofs with identities

The identity symbol is $\equiv$ and it means 'equivalent to'. For example, $x^2 - y^2 \equiv (x - y)(x + y)$.

To prove an identity you should start with the expression on one side of the identity symbol and manipulate it algebraically to eventually get the expression on the other side of the identity symbol. You need to show every step of your manipulation.

Example

Prove that $(x + y)^2 - (x - y)^2 \equiv 4xy$.

Proof. We will start with the left hand side as it seems more logical.

$$\begin{aligned} (x+y)^2 - (x-y)^2 &\equiv (x^2 + 2xy + y^2) - (x^2 - 2xy + y^2) \\ &\equiv (x^2 + 2xy + y^2) - x^2 + 2xy - y^2 \\ &\equiv 4xy \end{aligned}$$

So we have proved that $(x + y)^2 - (x - y)^2 \equiv 4xy$. □

2.3 Proofs with quadratics

The main two proofs involving quadratics are proofs involving information with how many roots the quadratic has and proofs involving inequalities.

Example

The equation $x^2 + (k - 3)x + (3 - 2k) = 0$, where k is a constant has no real roots. Prove that k satisfies $-3 < k < 1$.

Proof. In this proof we need to consider what information we have been given and go from there. We have been given a quadratic and the information that the quadratic has no real roots. No real roots indicates that the discriminant of the quadratic is less than 0, i.e. $b^2 - 4ac < 0$.

In our quadratic, $a = 1$, $b = k - 3$ and $c = 3 - 2k$ so

$$\begin{aligned} b^2 - 4ac &= (k-3)^2 - 4 \times 1 \times (3 - 2k) \\ &= (k-3)^2 - 4(3-2k) \\ &= (k-3)(k-3) - 4(3-2k) \\ &= k^2 - 6k + 9 - 12 + 8k \\ &= k^2 + 2k - 3 \end{aligned}$$

$b^2 - 4ac < 0$ so $k^2 + 2k - 3 < 0$ which is a quadratic inequality and we know how to solve these. We factorise to get $(k+3)(k-1) < 0$ which tells us that the quadratic crosses the x axis at -3 and 1. You should now draw a graph and ask yourself where is the quadratic less than 0? It is when $-3 < k < 1$ which concludes our proof. □

The most common proof to be seen in exams is one that involves a quadratic and an inequality symbol. These are extremely easy if you just remember to complete the square as we shall see in the next three examples.

Example

Prove that $3x^2 + 12x + 16 > 0$ for all values of x.

Proof. This question is essentially asking us to prove that this quadratic always outputs a value that is larger than 0 no matter what value of x we substitute into it. If we complete the square, then we will be able to clearly see why this is the case. Let's do that.

$$\begin{aligned} 3x^2 + 12x + 16 &= 3(x^2 + 4x) + 16 \\ &= 3((x+2)^2 - 4) + 16 \\ &= 3(x+2)^2 - 12 + 16 \\ &= 3(x+2)^2 + 4 \end{aligned}$$

Our question is now about proving that $3(x+2)^2 + 4 > 0$. First of all $(x+2)^2 \geq 0$ as any number squared is larger than or equal to 0. This means that $3(x+2)^2 \geq 0$. Finally we can add 4 to both sides to get $3(x+2)^2 + 4 \geq 4$ which proves that $3(x+2)^2 + 4$ is always greater than 0. □

Example

Prove that $x^2 + 8x + 20 \geq 4$ for all values of x.

Proof. This question is essentially asking us to prove that the quadratic $x^2 + 8x + 20$ always outputs a value that is larger than or equal to 4 no matter what value of x we substitute into it. This is where students slip up and start doing weird algebraic manipulations. If we want to prove that $x^2 + 8x + 20 \geq 4$, then this is equivalent to proving that $x^2 + 8x + 16 \geq 0$ which is easy to do as we just complete the square.

$$x^2 + 8x + 16 = (x+4)^2$$

Our question is now about proving that $(x+4)^2 \geq 0$ which is simpler than the last example. $(x+4)^2 \geq 0$ as any number squared is larger than or equal to 0. □

Example

Prove that $(x+5)^2 > 4x+9$ for all real values of x.

Proof. This question is essentially asking us to prove that the quadratic $(x+5)^2$ always outputs a value that is larger than $4x+9$ no matter what value of x we substitute in. This is where students slip up and start doing weird algebraic manipulations. If we want to prove that $(x+5)^2 > 4x+9$, then this is equivalent to proving that $(x+5)^2 - 4x - 9 > 0$ which is easy to do as we simplify it and then complete the square.

$$\begin{aligned}(x+5)^2 - 4x - 9 &= x^2 + 10x + 25 - 4x - 9 \\ &= x^2 + 6x + 16 \\ &= (x+3)^2 - 9 + 16 \\ &= (x+3)^2 + 7\end{aligned}$$

Our question is now about proving that $(x+3)^2 + 7 > 0$. $(x+3)^2 \geq 0$ as any number squared is larger than or equal to 0. Adding 7 to both sides gets us $(x+3)^2 + 7 \geq 7$. So $(x+3)^2 + 7$ is larger than 0 no matter what value of x you substitute into it. □

2.4 Proofs with circles

Example

A circle has equation $(x-1)^2 + y^2 = k$, where $k > 0$. The straight line L with equation $y = px$ cuts the circle at two distinct points. Prove that $k > \frac{p^2}{1+p^2}$.

Proof. Let us start with the information that we are given. The information that the straight line L with equation $y = px$ cuts the circle at two distinct points means that there will be two solutions for x and two solutions for y if we were to solve the simultaneous equations $(x-1)^2 + y^2 = k$ and $y = px$. Let us start to run through the process of solving them and then go from there.

Substituting L into $(x-1)^2 + y^2 = k$ we get

$$\begin{aligned}(x-1)^2 + (px)^2 &= k \\ x^2 - 2x + 1 + p^2x^2 &= k \\ (1+p^2)x^2 - 2x + 1 &= k \\ (1+p^2)x^2 - 2x + (1-k) &= 0\end{aligned}$$

The line L cuts the circle at two distinct points so there must be two solutions to this quadratic equation which means that the discriminant must be larger than 0.

In our quadratic, $a = (1+p^2)$, $b = -2$ and $c = 1-k$ so

$$\begin{aligned}b^2 - 4ac &= (-2)^2 - 4 \times (1+p^2) \times (1-k) \\ &= 4 - 4(1+p^2)(1-k) \\ &= 4 - 4(1 - k + p^2 - kp^2) \\ &= 4 - 4 + 4k - 4p^2 + 4kp^2 \\ &= 4k - 4p^2 + 4kp^2 \\ &= 4k + (4k-4)p^2\end{aligned}$$

$b^2 - 4ac > 0$ so $4k + (4k-4)p^2 > 0$ which is a linear inequality in terms of k and we know how to solve these. We need to rearrange it to get our required result.

$$\begin{aligned}
4k + (4k-4)p^2 &> 0 \\
k + (k-1)p^2 &> 0 \\
k + kp^2 - p^2 &> 0 \\
k + kp^2 &> p^2 \\
k(1+p^2) &> p^2 \\
k &> \frac{p^2}{1+p^2}
\end{aligned}$$

which is our required result. □

2.5 Proofs with 'if' statements

Example

Prove that if $(2+3x)^3 > 80 + 54x^2 + 27x^3$, then $x > 2$.

It is important to note what this question means. It is saying prove that IF this happens, THEN this happens. This means that you start with the given information of $(2+3x)^3 > 80 + 54x^2 + 27x^3$ and manipulate this to get the required result of $x > 2$.

Proof.

$$\begin{aligned}
(2+3x)^3 &> 80 + 54x^2 + 27x^3 \\
(2+3x)(2+3x)(2+3x) &> 80 + 54x^2 + 27x^3 \\
(4+12x+9x^2)(2+3x) &> 80 + 54x^2 + 27x^3 \\
8 + 12x + 24x + 36x^2 + 18x^2 + 27x^3 &> 80 + 54x^2 + 27x^3 \\
8 + 36x + 54x^2 + 27x^3 &> 80 + 54x^2 + 27x^3 \\
36x &> 72 \\
x &> 2
\end{aligned}$$

as required. □

2.6 Geometrical proofs

Example

Prove that the points $A(-2,3)$, $B(2,1)$ and $C(14,-5)$ are collinear.

You may not have come across the word collinear before. If A, B and C lie on a straight line, then they are said to be collinear.

Proof. If we prove that the line segments AB and BC have the same gradient and observe that they have a common point B, then the three points must lie on the same straight line, i.e. collinear.

The gradient of AB is $\frac{1-3}{2--2} = \frac{-2}{4} = -\frac{1}{2}$.

The gradient of BC is $\frac{-5-1}{14-2} = \frac{-6}{12} = -\frac{1}{2}$.

The line segments AB and BC have the same gradient and share the common point B. Therefore, the points A, B and C are collinear. □

Other geometrical proofs include proving that three points are the vertices of a right angled triangle. An example of this is given on page 148 of the Pearson Year 1 Pure textbook.

3 Proof by exhaustion

Proof by exhaustion is literally what it says on the tin. It is exhausting. The proof will involve you proving that a statement works for a limited amount of cases.

Example

Prove that $(2n+1)^2 > 2^{n+1}$ for all positive integers less than 7.

Proof. We need to prove this statement for all positive whole numbers from 1 to 6.

When $n = 1$,

$$\begin{aligned}(2\times 1+1)^2 &> 2^{1+1}\\ 3^2 &> 2^2\\ 9 &> 4\end{aligned}$$

which is true.

When $n = 2$,

$$\begin{aligned}(2\times 2+1)^2 &> 2^{2+1}\\ 5^2 &> 2^3\\ 25 &> 8\end{aligned}$$

which is true.

When $n = 3$,

$$\begin{aligned}(2\times 3+1)^2 &> 2^{3+1}\\ 7^2 &> 2^4\\ 49 &> 16\end{aligned}$$

which is true.

When $n = 4$,

$$\begin{aligned}(2\times 4+1)^2 &> 2^{4+1}\\ 9^2 &> 2^5\\ 81 &> 32\end{aligned}$$

which is true.

When $n = 5$,

$$\begin{aligned}(2 \times 5 + 1)^2 &> 2^{5+1}\\ 11^2 &> 2^6\\ 121 &> 64\end{aligned}$$

which is true.

When $n = 6$,

$$\begin{aligned}(2 \times 6 + 1)^2 &> 2^{6+1}\\ 13^2 &> 2^7\\ 169 &> 128\end{aligned}$$

which is true.

Since this statement is true for all positive integers between 1 and 6 we have proved the conjecture. $\square$

4 Proof by the use of jottings

This is a big one because students never know when to use it. It involves an inequality and TWO variables like x and y. Please note that if you have one variable, then you use the completing the square method mentioned in section 2.3.

Example

Prove that for any positive numbers x and y, $x + y \geq \sqrt{4xy}$.

We need to use jottings here and the reason is that we have two variables (x and y) and an inequality. The reason for jottings is that we have no idea how to go about proving that $x + y \geq \sqrt{4xy}$. We need to start our proof with something that we know is true. Jottings involves starting with $x + y \geq \sqrt{4xy}$ and working our way back to something that we can start with in our proof that we know is true.

Jottings

$$\begin{aligned}x + y &\geq \sqrt{4xy}\\ (x + y)^2 &\geq 4xy\\ x^2 + 2xy + y^2 &\geq 4xy\\ x^2 - 2xy + y^2 &\geq 0\\ (x - y)^2 &\geq 0\end{aligned}$$

which we know is always true because any number squared is larger than or equal to 0. Please note that I was only able to square both sides of the inequality on line 2 because both x and y are positive which meant that both sides were positive. For example, $4 > 3$ and so squaring results in $16 > 9$

which is still true. However, $-2 > -3$ and squaring results in $4 > 9$ which is not true. Both sides of the inequality must be positive if you want to square both sides.

Proof. We observe that the expression $(x-y)^2$ is larger than or equal to 0 as any number squared is larger than or equal to 0.

$$
\begin{aligned}
(x-y)^2 &\geq 0 \\
x^2 - 2xy + y^2 &\geq 0 \\
x^2 + 2xy + y^2 &\geq 4xy \\
(x+y)^2 &\geq 4xy \\
x + y &\geq \sqrt{4xy} \text{ which can be done due to } x \text{ and } y \text{ being positive.}
\end{aligned}
$$

This proves that for any positive integers x and y, we have that $x + y \geq \sqrt{4xy}$. □

5 Proof by counter-example

Proof by counter-example is used to prove that a conjecture is not true. A counter-example is one example where the conjecture does not work and so disproves the conjecture all together.

I am going to do another jottings example which is then followed with a question on using a counter-example as jottings can be difficult to grasp.

Example

Prove that for any positive numbers x and y, $\frac{x}{y} + \frac{y}{x} \geq 2$.

We need to use jottings here and the reason is that we have two variables (x and y) and an inequality. The reason for jottings is that we have no idea how to go about proving that $\frac{x}{y} + \frac{y}{x} \geq 2$. We need to start our proof with something that we know is true. Jottings involves starting with $\frac{x}{y} + \frac{y}{x} \geq 2$ and working our way back to something that we can start with in our proof that we know is true.

Jottings

$$
\begin{aligned}
\frac{x}{y} + \frac{y}{x} &\geq 2 \\
\frac{x^2}{xy} + \frac{y^2}{xy} &\geq 2 \\
\frac{x^2 + y^2}{xy} &\geq 2 \\
x^2 + y^2 &\geq 2xy \\
x^2 - 2xy + y^2 &\geq 0 \\
(x-y)^2 &\geq 0
\end{aligned}
$$

which we know is always true because any number squared is larger than or equal to 0. Please note that the inequality sign did not flip in line 4 due to x and y being positive and so xy is positive.

Proof. We observe that the expression $(x-y)^2$ is larger than or equal to 0 as any number squared is larger than or equal to 0.

$$
\begin{aligned}
(x-y)^2 &\geq 0 \\
x^2 - 2xy + y^2 &\geq 0 \\
x^2 + y^2 &\geq 2xy \\
\frac{x^2+y^2}{xy} &\geq 2 \\
\frac{x^2}{xy} + \frac{y^2}{xy} &\geq 2 \\
\frac{x}{y} + \frac{y}{x} &\geq 2
\end{aligned}
$$

This proves that for any positive integers x and y, we have that $\frac{x}{y} + \frac{y}{x} \geq 2$. □

Example

Use a counter-example to prove that $\frac{x}{y} + \frac{y}{x} \geq 2$ is not always true when x and y are not both positive.

Proof. Let $x = 1$ and $y = -1$ so that we have $\frac{1}{-1} + \frac{-1}{1} = -1 + -1 = -2$ which is not greater than or equal to 2. So the statement $\frac{x}{y} + \frac{y}{x} \geq 2$ is not always true when x and y are not both positive. □